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The Structure of Multiloop Amplitudes in Gauge and Gravity Theories

Z. Bern[a*], J. J. M. Carrasco[a] and H. Johansson[b]

[a]Department of Physics and Astronomy, UCLA, Los Angeles, CA 90095-1547, USA

[b]Institut de Physique Théorique, CEA–Saclay, F–91191 Gif-sur-Yvette cedex, France

We review the recently discovered duality between color and kinematics in gauge theories. This duality leads to a remarkably simple double-copy relation between diagrammatic numerators of gravity scattering amplitudes and gauge-theory ones. We summarize nontrivial evidence that the duality and double-copy property holds to all loop orders. We also comment on other developments, including a proof that the gauge-theory duality leads to the gravity double-copy property, and the identification of gauge-theory Lagrangians whose double copies yield gravity Lagrangians.

1. Introduction

Gauge theory and gravity scattering amplitudes have a far richer structure than evident from their respective Lagrangians. As one such example, in this talk we will describe a recently discovered duality between color and kinematic numerators of gauge-theory scattering amplitude diagrams [1,2]. Remarkably, this duality appears to have important implications for gravity: when the gauge-theory numerators satisfy the duality, the numerators of corresponding gravity theories are given by a double copy of the gauge theory numerators, diagram by diagram [1], as demonstrated recently [3]. The double-copy property has the benefit of greatly clarifying the mysterious Kawai, Lewellen and Tye (KLT) relation between gauge and gravity tree amplitudes [4]. In this talk we will focus on the recent progress in extending the duality and double-copy properties to loop level [2]. We also summarize the structure of gauge-theory Lagrangians whose Feynman diagrams satisfy the duality, leading to gravity Lagrangians that exhibit the double-copy property [3].

Many of the recent developments for scattering amplitudes rely on on-shell methods, which include on-shell recursion [5] at tree level and the unitarity method [6] at loop level. A particularly powerful approach to generalized unitarity cut constructions [7] is the method of maximal cuts [8,9], organizing the calculation starting from unitarity cuts where all propagators are cut and then systematically reducing the number of cut propagators. On-shell methods allow one to construct new amplitudes using simpler on-shell amplitudes as input. Cuts that decompose loop amplitudes into tree amplitudes, such as the sample four-loop ones displayed in fig. 1, are generally the most advantageous ones to use. The unitarity method gives a set of rules for reconstructing complete amplitudes from cuts. As discussed in a variety of talks at this conference [10], the unitarity method is by now a standard tool in loop computations. As one state-of-the-art example from collider physics, it plays a central role in progress towards the long-awaited NLO calculation of $W + 4$-jet production at the LHC, described in this conference [11].

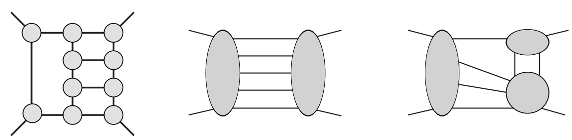

Figure 1. Some examples of generalized cuts at four loops which decompose loop amplitudes into sums of products of tree amplitudes.

Multiloop calculations are typically rather involved, yielding expressions with no easily acces-

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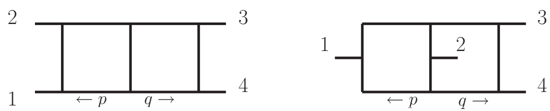

Figure 2. The two-loop QCD amplitudes identical helicity amplitudes can be expressed in terms of planar and non-planar double-box integrals, as well as a "bow-tie" integral (not displayed here).

sible structure or pattern. Consider, *e.g.,* the rather lengthy results for $gg \to gg$ scattering amplitudes in QCD, given in refs. [12]. Might there be a structure hidden in these rather opaque expressions? To answer this question, it is easiest to begin by looking at one the simplest $2 \to 2$ two-loop amplitudes—identical helicity scattering in QCD—first computed in ref. [13].

The all-plus helicity amplitude is given in terms of three integrals, two of which are displayed in fig. 2. The planar and non-planar double-box contributions shown in fig. 2 may be expressed in terms of loop integrals as in ref. [13],

$$A^{\rm P}_{1234} = i\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\,s_{12}\,I_4^{\rm P}\Big[N(p,q)\Big]\,,$$
$$A^{\rm NP}_{12;34} = i\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\,s_{12}\,I_4^{\rm NP}\Big[N(p,q)\Big]\,,$$

where the $N(p,q)$ represent numerator polynomials depending on the loop momenta p and q. Remarkably, the polynomials (and the prefactors) for both integrals are identical. The shared numerator is given by,

$$N(p,q) = (D_s-2)(\lambda_p^2\,\lambda_q^2 + \lambda_p^2\,\lambda_{p+q}^2 + \lambda_q^2\,\lambda_{p+q}^2) + 16\Big((\lambda_p\cdot\lambda_q)^2 - \lambda_p^2\,\lambda_q^2\Big)\,,$$

where the vectors $\vec{\lambda}_i$ represent the (-2ϵ)-dimensional components of the loop momenta ℓ_i; that is, $\ell_i \equiv \ell_i^{[4]} + \lambda_i$, where $\ell_i^{[4]}$ has only four-dimensional components and we take (-2ϵ) to be positive. The propagators of the integrals are of course different and correspond to the two diagrams in fig. 2. The prefactor depends on spinor inner products $\langle i\,j\rangle$ and $[i\,j]$; further details on spinor-helicity notation may be found in, for example, ref. [14]. We denote the number of gluon states circulating in the loop by D_s-2, where D_s is the dimension of spacetime. (D_s is the dimension of the polarization vectors which can be taken to be independent of the dimension of the loop momenta.)

The above result has a rather striking feature: The numerators of the planar and non-planar integrals are identical. This feature is rather obscure from a Feynman diagram point of view, where no discernible relationship between the planar and non-planar contributions is apparent. This curious feature seems to be an important clue for novel structures in gauge-theory amplitudes. But what might it be? We know now [1,2] that this curiosity is a hint of hidden structures, including:

- A new duality between color and kinematics.
- All-loop relations between planar and non-planar diagrammatic integrands.
- A double-copy structure of gravity diagram kinematic numerators in terms of gauge-theory ones.

Although these properties are not yet completely proven, below we discuss evidence that these properties hold to all loop orders.

2. A duality between color and kinematics

Any amplitude with all particles in the adjoint representation of the gauge group can be arranged into the form,

$$\mathcal{A}_n^{\rm tree}(1,2,3,\ldots,n) = \sum_i \frac{n_i\,c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}\,, \qquad (1)$$

where the sum runs over the set of n-point diagrams with only cubic vertices and we suppress factors of the coupling constant. The product runs over all propagators (internal lines) $1/p^2_{\alpha_i}$ of each diagram. The c_i are the color factors obtained by dressing every three vertex with an $\widetilde{f}^{abc} = i\sqrt{2}f^{abc}$ structure constant, and the n_i are kinematic numerator factors depending on momenta, polarizations and spinors (and Grassmann parameters for supersymmetric amplitudes expressed in superspace). The form (1) follows

trivially from Feynman diagrams, by representing all contact terms as inverse propagators in the kinematic numerators, which then cancel propagators.

2.1. Tree-level duality

According to the duality proposal of ref. [1], arrangements of the diagrammatic numerators in eq. (1) exist such that they satisfy equations in one-to-one correspondence with the color Jacobi identities. Specifically, we demand that every color Jacobi identity induces a kinematic identity:

$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k \,. \tag{2}$$

This duality is expected to hold in a large variety of theories, including supersymmetric extensions of Yang-Mills theory. At four points the duality is automatically satisfied for any choice of valid numerators [15,1], but at higher points the existence of such arrangements is rather nontrivial. Indeed there is, as of yet, no complete proof that this can always be accomplished. Surprisingly, this duality implies non-trivial relations between the color-ordered partial amplitudes of gauge theory [1]. A proof of these amplitude relations has recently been given both in field theory and in string theory [16,17].

2.2. Gravity as a double copy of gauge theory

Remarkably, once the gauge-theory amplitudes are arranged into a form satisfying the duality (2), gravity tree amplitudes are given by a double copy of gauge-theory numerator factors [1],

$$-\,i\mathcal{M}_n^{\rm tree}(1,2,\ldots,n) = \sum_i \frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}\,, \tag{3}$$

where the $\tilde{n}_i$ represent numerator factors of a second gauge theory amplitude, the sum runs over the same set of diagrams as in eq. (1), and we suppressed the gravitational coupling constant. This formula recently has been proven for pure gravity and for $\mathcal{N}=8$ supergravity, under the assumption that the duality (2) holds in the corresponding gauge theories, by use of on-shell recursion [3]. If pure-Yang-Mills amplitudes are used, the obtained gravity amplitudes correspond to Einstein gravity coupled to an antisymmetric tensor and dilaton; the n-graviton tree-level amplitudes of this theory are those of pure Einstein gravity. If both families of kinematic factors are for the $\mathcal{N}=4$ super-Yang-Mills theory, the gravity theory amplitudes are those of $\mathcal{N}=8$ supergravity. For this double-copy relation to hold, at least one family of numerators (n_i or $\tilde{n}_i$) needs to satisfy the duality (2) [1,3]. Additionally, the $\tilde{n}_i$ numerator factors need not come from the same theory as the n_i factors. This allows for the construction of gravity amplitudes with varying amounts of supersymmetry. The duality has also been studied in string theory [18]. We note that the heterotic string offers a natural venue for understanding these properties because of the parallel treatment of color and kinematics [19].

2.3. Lagrangian insight

If one compares the Yang-Mills and Einstein gravity Lagrangians

$$\mathcal{L}^{\rm YM} = \frac{1}{4g^2}F^2\,, \qquad\qquad \mathcal{L}^{\rm G} = \frac{2}{\kappa^2}\sqrt{-g}R\,, \tag{4}$$

it seems rather puzzling that gravity can be a double copy of gauge theory. Indeed, the perturbative expansions appear to be completely different: expanding the metric around flat space, we find that gravity is composed of an infinite sequence of increasingly complicated vertices, whereas the Yang-Mills Lagrangian has local interactions that terminate at four points. On the other hand, the KLT relations map the tree-level scattering amplitudes of gravity to those of gauge theory. This suggests that the standard Lagrangians (4) obscure the relation between the two.

Some initial steps in understanding the relationship between gravity and gauge theory at the level of the Lagrangian were carried out in ref. [20]. In particular, that paper developed some tricks for separating the graviton Lorentz indices into “left” and “right” classes, consistent with the double-copy property. (See also ref. [21].) However, the connection to gauge theory remained rather mysterious. As realized in ref. [3], the key missing ingredient was the color-kinematic duality of ref. [1].

To see how to make the proper rearrangement, consider first the Yang-Mills Lagrangian. The

initial step is to rearrange the Lagrangian so its Feynman diagrams satisfy the color-kinematic duality. As noted earlier for four-point amplitudes, the duality is automatic—the ordinary form of the Yang-Mills Lagrangian generates diagrams that satisfy the duality. Beyond four points, for the duality to be satisfied directly by the Feynman diagrams, we must modify the Lagrangian yet leave the amplitudes unchanged. For five points, this may be accomplished by adding the following term to the Yang-Mills Lagrangian [3]:

$$\begin{aligned}\mathcal{L}'_5 \;=\; -\frac{1}{2}g^3(f^{a_1a_2b}f^{ba_3c} \\ +f^{a_2a_3b}f^{ba_1c}+f^{a_3a_1b}f^{ba_2c})f^{ca_4a_5} \\ \times\partial_{[\mu}A^{a_1}_{\nu]}A^{a_2}_{\rho}A^{a_3\mu}\frac{1}{\Box}(A^{a_4\nu}A^{a_5\rho})\,. \qquad (5)\end{aligned}$$

This new term leaves the amplitudes unchanged because it vanishes identically by the color-Jacobi identity. Nevertheless, the canceling terms have different color structures and thus are associated with different diagrams. These terms modify the individual diagrams so that the duality holds. The additional term (5) is, however, nonlocal. We can make the Lagrangian local by introducing a set of auxiliary fields as explained in ref. [3]. In fact, with a sufficient number of such fields, all higher-point interactions can be replaced by three-point interactions. Once the gauge-theory Lagrangian has been put into this form, a gravity Lagrangian which yields the correct amplitudes is given simply by two copies of the gauge theory Lagrangian, as described in ref. [3], with the identification $A^a_\mu\tilde{A}^b_\nu \to h_{\mu\nu}$, dropping the color factors. (The color indices play essentially no role in this identification.)

If one had an all orders Lagrangian, it would be possible to investigate non-perturbative implications. Because of the double-copy property, one might expect that *all* classical solutions of gravity could be written as double copies of solutions of gauge theories. In particular, in coordinate space we can expect gravity solutions to be convolutions of gauge-theory solutions, with the schematic form, $g_{\mu\nu}(x) \sim \int d^Dy\, A^a_\mu(x-y)\tilde{A}^b_\nu(y)$.

3. The color-kinematic duality at loop level

Very recently ref. [2] proposed that the color-kinematic duality and gravity double-copy property extends to *all* loop orders. This might seem a rather bold conjecture, yet the unitarity method provides strong motivation. Indeed the duality was first observed in the three- and four-loop four-point amplitudes of $\mathcal{N}=4$ super-Yang-Mills theory, with various on-shell conditions imposed [1]. The success of the duality at the Lagrangian level [3] is also quite suggestive.

The loop-level expressions for gauge and gravity amplitudes would then be,

$$\begin{aligned}(-i)^L\mathcal{A}^{\rm loop}_n &= \sum_j\int\prod_{l=1}^{L}\frac{d^Dp_l}{(2\pi)^D}\frac{1}{S_j}\frac{n_jc_j}{\prod_{\alpha_j}p^2_{\alpha_j}}\,, \qquad (6)\\ (-i)^{L+1}\mathcal{M}^{\rm loop}_n &= \sum_j\int\prod_{l=1}^{L}\frac{d^Dp_l}{(2\pi)^D}\frac{1}{S_j}\frac{n_j\tilde{n}_j}{\prod_{\alpha_j}p^2_{\alpha_j}}\,,\end{aligned}$$

where we again suppressed the couplings and the sums now run over all distinct n-point L-loop diagrams with cubic vertices. These include distinct permutations of external legs, and the S_j are the symmetry factors of each diagram. As at tree level, at least one family of numerators (n_j or $\tilde{n}_j$) is constrained to satisfy the duality (2). We expect these formulas to hold in a large class of theories, including theories which are the low energy limits of string theories. It should also hold in pure gravity, but in this case extra projectors would be required to remove the extra unwanted states arising in the direct product of two pure Yang-Mills theories.

As at tree level, the ability to organize amplitudes around cubic graphs is trivially accomplished by inserting inverse propagators into the numerators to account for contact terms. The non-trivial part of this conjecture is that there always exists sufficient freedom to arrange gauge-theory multiloop amplitudes in a way that satisfies the color-kinematic duality (2). The unitarity method straightforwardly ensures that the double-copy property of gravity extends to loop level when it holds at tree level, since all cuts that decompose loop amplitudes into products of tree amplitudes will have this property [2,3].

3.1. Checks on the loop-level conjecture

Since we do not yet have a proof that the color-kinematic duality holds, it is important to check some explicit examples. The known one and two-loop four-point amplitudes of $\mathcal{N}=4$ super-Yang-Mills theory and $\mathcal{N}=8$ supergravity, as given in ref. [22], satisfy the conjectures (6). Another example is the previously mentioned two-loop four-point identical-helicity amplitude of pure Yang-Mills theory [13], which can also be shown to satisfy the duality [1,2].

A more sophisticated example, which we outline here, is the three-loop four-point amplitude of $\mathcal{N}=4$ super-Yang-Mills theory. This amplitude offers a rather non-trivial check of the duality given both the high loop order and the non-trivial dependence of the numerators on loop momenta. It was originally constructed in refs. [23,9] via the unitarity method. In ref. [2], the duality was made manifest by appropriate reshufflings of terms in the earlier forms of the amplitude.

The duality requires that the numerator identities in eq. (2) must be imposed for every propagator in every diagram. In fig. 4 we display one such numerator relation. For any diagram, we can describe any internal line, carrying some momentum l_s, in terms of formal graph vertices $V(p_a,p_b,l_s)$, and $V(-l_s,p_c,p_d)$ where the p_i are the momenta of the other legs attached to l_s, as illustrated on the left side of fig. 4. The duality (2) requires that,

$$
\begin{aligned}
n(\{\,V(p_a,p_b,l_s),\,V(-l_s,p_c,p_d),\,\cdots\}) = \\
n(\{\,V(p_d,p_a,l_t),\,V(-l_t,p_b,p_c),\,\cdots\}) \\
+\,n(\{\,V(p_a,p_c,l_u),\,V(-l_u,p_b,p_d),\,\cdots\})\,,
\end{aligned}
$$

where n represents the numerator associated with the diagram specified by the set of vertices, the omitted vertices are identical in all three diagrams, and $l_s \equiv (p_c+p_d)$, $l_t \equiv (p_b+p_c)$ and $l_u \equiv (p_b+p_d)$ in the numerator expressions. There is one such equation for every propagator in every diagram. Solving the system of distinct equations enforces the duality conditions (2). It turns out[2] that a solution to the duality relations is found in terms of the 12 diagrams displayed in fig. 3.

Of course the amplitude must be the correct one for the theory under consideration, and thus must satisfy all cut conditions as well. The explicit solution found in ref. [2], satisfying both the cut conditions and the duality, is displayed in table 1. Interestingly, after imposing the duality, the maximal cut of diagram (e) in fig. 3 is sufficient for finding the unique solution, when the numerators are local and no diagram has a worse power counting than the known amplitude. In this table, an overall factor of $stA_4^{\rm tree}$ has been removed from the entries, s,t,u are Mandelstam invariants corresponding to $(k_1+k_2)^2, (k_2+k_3)^2, (k_1+k_3)^2$ and $\tau_{ij}=2k_i\cdot l_j$, where k_i and

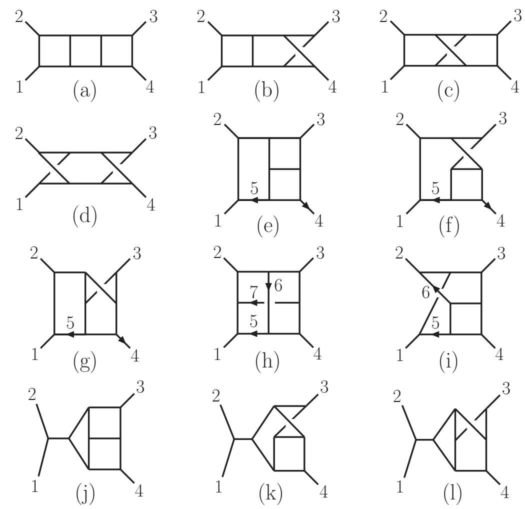

Figure 3. The 12 diagrams contributing to both $\mathcal{N}=4$ super-Yang-Mills and $\mathcal{N}=8$ supergravity three-loop four-point amplitudes in the arrangement of ref. [2]. The corresponding integrals are obtained by combining their propagators with numerator factors given in table 1. The (internal) symmetry factor for diagram (d) is $S_{\rm (d)}=2$, the rest are unity. All distinct external permutations of each diagram appear in the amplitude.

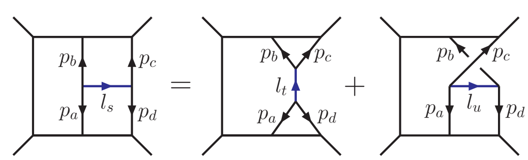

Figure 4. A numerator duality relation at three loops.

Table 1
The numerator factors of the integrals in fig. 3. The first column labels the integral, the second column the relative numerator factor for $\mathcal{N}=4$ super-Yang-Mills theory. The square of this is the relative numerator factor for $\mathcal{N}=8$ supergravity.

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$[s\,(-\tau_{35}+\tau_{45}+t)-t\,(\tau_{25}+\tau_{45})+u\,(\tau_{25}+\tau_{35})-s^2]/3$
(h)	$[s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+\,t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2]/3$
(i)	$[s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+\,t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2]/3$
(j)-(l)	$s(t-u)/3$

l_j are external and internal momenta as labeled in fig. 3. The reader may check that all duality relations hold. Ref. [2] also verified that the double-copy property holds for $\mathcal{N}=8$ supergravity, confirming it on a complete set of cuts using the known amplitude [23,9].

4. Comments and conclusions

In this talk we summarized a recently proposed gauge-theory duality between color and kinematics, leading to a double-copy property for gravity theories [1,2]. Although the duality remains a conjecture, we can even now exploit it to guide loop computations, simply by enforcing the duality and verifying the consistency with the unitarity cuts. This is helpful both for organizing gauge-theory amplitudes and for obtaining the corresponding double-copy gravity amplitudes.

There are a number of interesting open problems related to the color-kinematic duality. In particular, it would be helpful to carry out further checks of the duality for multiloop processes. More generally an all-orders proof would be important, especially if it leads to new insight into the origins of the duality. The duality suggests the existence of a group theoretical construction of the kinematic numerators, which would, of course, be very interesting to develop. We would also like to have Lagrangians whose diagrams satisfy the duality to all orders, and which give gravity Lagrangians as double copies, along the lines described in ref. [3]. We expect this to have non-trivial implications at strong coupling. It seems likely that the duality should hold as well in higher-genus perturbative string theory.

The duality and double-copy property may also shed new light on the issue of ultraviolet divergences in $\mathcal{N}=8$ supergravity. For four-point amplitudes through four loops, explicit computations show that ultraviolet cancellations exist beyond those needed for finiteness [22,23,9,24]. Beyond this, the situation is less clear. A consensus has formed that supersymmetry alone cannot prevent divergences in four dimensions starting at seven loops [25,26,27].[2] Although nontrivial cancellations are known to exist to all loop orders [29], we do not know if these cancellations are sufficient to render the theory finite. Recent reviews discussing the ultraviolet properties of $\mathcal{N}=8$ supergravity may be found in refs. [30].

The color-kinematic duality and gravity double-copy structure likely have important nonperturbative implications. In particular, these properties suggest that classical solutions in gravity theories may be expressible as double copies of classical solutions in gauge theories.

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[2]The claimed delay from supersymmetry of potential ultraviolet divergences in $\mathcal{N}=8$ supergravity until nine loops [28] has been retracted [27].

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